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Rapports de Recherche

N° 1222

Programme 5
Automatique, Productique,
Traitement du Signal et des Données

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Mai 1990

RR.1222

TIME–DISCRETIZATION OF THE ZAKAI EQUATION FOR DIFFUSION PROCESSES OBSERVED IN CORRELATED NOISE*

Discrétisation en temps de l'équation de Zakai
pour des processus de diffusion observés avec un bruit corrélé

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*Research partially supported by USACCE under Contract DAJA45–90–C–0008.
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Résumé

On propose un schéma de discrétisation en temps pour l'équation de Zakai, une EDP stochastique qui fournit la loi conditionnelle d'un processus de diffusion observé avec un bruit blanc additif. On considère le cas où les bruits d'observation et d'état sont corrélés. Ce schéma numérique repose sur une formule produit à la Trotter, qui permet de mettre en évidence des étapes de *prédiction* et de *correction*, et pour laquelle on établit une estimation d'erreur d'ordre δ, où δ est le pas de discrétisation en temps. A l'étape de *correction* est associée une EDP stochastique du second ordre dégénérée, pour laquelle un résultat de représentation à l'aide de courbes caractéristiques stochastiques a été prouvé par Krylov–Rozovskii [8] et Kunita [9,11]. Finalement, un schéma de discrétisation est proposé pour approcher ces courbes caractéristiques stochastiques. Sous une hypothèse supplémentaire portant sur le coefficient de corrélation, on établit une estimation d'erreur d'ordre $\sqrt{\delta}$ pour le schéma global. Cette estimation est la meilleure possible d'après Elliott–Glowinski [5].

Mots–Clés: *processus de diffusion, bruits corrélés, filtrage non linéaire, équation de Zakai, EDP stochastiques, caractéristiques stochastiques, discrétisation en temps*

Abstract

A time discretization scheme is provided for the Zakai equation, a stochastic PDE which gives the conditional law of a diffusion process observed in white–noise. The case where the observation noise and the state noise are correlated, is considered. The numerical scheme is based on a Trotter–like product formula, which exhibits *prediction* and *correction* steps, and for which an error estimate of order δ is proved, where δ is the time discretization step. The *correction* step is associated with a degenerate second–order stochastic PDE, for which a representation result in terms of stochastic characteristics has been proved by Krylov–Rozovskii [8] and Kunita [9,11]. A discretization scheme is then provided to approximate these stochastic characteristics. Under an additional assumption on the correlation coefficient, an error estimate of order $\sqrt{\delta}$ is proved for the overall numerical scheme. This has been proved to be the best possible error estimate by Elliott–Glowinski [5].

Keywords: *diffusion processes, correlated noises, nonlinear filtering, Zakai equation, stochastic PDE, stochastic characteristics, time discretization*

Contents

1 Introduction

The purpose of this paper is to present a *computable* time discretization scheme for the Zakai equation of nonlinear filtering with correlated noises, and to provide an estimate of the rate of convergence.

In the case of independent noises, the problem has been studied by Newton [13], Korezlioglu–Mazziotto [6], Bennaton [1], DiMasi–Pratelli–Runggaldier [4], Picard [14], Bensoussan–Glowinski–Rascanu [2] and LeGland [12]. Some of these authors have actually considered the associated Zakai equation. Time discretization schemes have been provided with a rate of convergence of order δ, where δ is the time discretization step.

In the case of correlated noises, the problem has been studied by Elliott–Glowinski [5]. The best approximation of the continuous filter based on the values of the observation process at a regular partition (with mesh δ) has been considered, and it has been proved that the rate of convergence is of order $\sqrt{\delta}$. However, no algorithm is provided to actually *compute* this approximation.

The paper is organized as follows. In Section 2, the nonlinear filtering problem is presented. Some results on the Zakai equation, and on a related degenerate second–order stochastic PDE, are recalled in Section 3. A Trotter–like product formula is then considered, with an error estimate of order δ. However, this numerical scheme is not *computable*. In Section 4, a representation result in terms of *stochastic characteristics* is presented for the degenerate second–order stochastic PDE. This part follows mainly the work of Krylov–Rozovskii [8] – see also Kunita [9,11]. Under an additional assumption on the correlation coefficient, a time discretization scheme is presented in Section 5 – based on an approximation of the stochastic characteristics – with an error estimate of order $\sqrt{\delta}$. In addition, this numerical scheme is actually *computable*, as far as time discretization is concerned.

2 The filtering problem

On the probability space $(\Omega, \mathcal{F}, P)$, consider the stochastic differential system

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t + \rho(X_t)\,dV_t$$

$$dY_t = h(X_t)\,dt + dV_t$$

where $\{W_t\,,\, t \geq 0\}$ and $\{V_t\,,\, t \geq 0\}$ are independent Wiener processes, with covariance matrix I (identity) and r respectively, and the random variable X_0 is independent of the Wiener processes, with probability law $p_0(x)\,dx$. For the clarity of exposition, it is assumed throughout the paper that $r = I$.

Throughout the paper, the coefficients will satisfy the following hypotheses

(i) b is a measurable and bounded function from $\mathbf{R}^m$ to $\mathbf{R}^m$,

(ii) σ is a continuous and bounded function on $\mathbf{R}^m$, with $a \triangleq \sigma\sigma^*$, such that

- *a is a uniformly elliptic $m \times m$ matrix, i.e. $a(x) \geq \alpha I$,*
- *$\overline{a}^i \triangleq \displaystyle\sum_{j=1}^m \frac{\partial a^{ij}}{\partial x_j}$ is a measurable and bounded function on $\mathbf{R}^m$,*

(iii) ρ is a continuous and bounded function from $\mathbf{R}^m$ to the space of $m \times d$ matrices, with $c \triangleq \rho\rho^*$, such that

- *$\alpha_k \triangleq \displaystyle\sum_{i=1}^m \frac{\partial \rho_k^i}{\partial x_i}$ is a measurable and bounded function on $\mathbf{R}^m$,*
- *$\overline{c}^i \triangleq \displaystyle\sum_{j=1}^m \frac{\partial c^{ij}}{\partial x_j}$ is a measurable and bounded function on $\mathbf{R}^m$,*

(iv) h is a measurable and bounded function from $\mathbf{R}^m$ to $\mathbf{R}^d$.

With the diffusion process $\{X_t\,,\, t \geq 0\}$ are associated the two partial differential operators

$$L \triangleq \tfrac{1}{2} \sum_{i,j=1}^m [a^{ij} + c^{ij}] \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^m b^i \frac{\partial}{\partial x_i}\,,$$
$$L_0 \triangleq \tfrac{1}{2} \sum_{i,j=1}^m a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^m b^i \frac{\partial}{\partial x_i}\,.$$

An other family of partial differential operators to be considered is

$$B_k \triangleq h_k + \sum_{i=1}^m \rho_k^i \frac{\partial}{\partial x_i}\,, \quad 1 \leq k \leq d\,.$$

Introducing

$$Z_t^s \triangleq \exp\left\{\int_s^t h^*(X_\tau)\,dY_\tau - \tfrac{1}{2}\int_s^t |h(X_\tau)|^2\,d\tau\right\}\ ,\qquad Z_t \triangleq Z_t^0\ ,$$

it is standard that, for all $T > 0$ the original probability measure P is equivalent on $[0, T]$ to the *reference probability* measure $P^\dagger$ with Radon–Nikodym derivative Z_T, so that under $P^\dagger$

$$dX_t = [b(X_t) - \rho(X_t)h(X_t)]\,dt + \sigma(X_t)\,dW_t + \rho(X_t)\,dY_t \tag{2.1}$$

where $\{W_t\,,\,t \geq 0\}$ and $\{Y_t\,,\,t \geq 0\}$ are independent Wiener processes, with covariance matrix I (identity), and the random variable X_0 is independent of the Wiener processes, with probability law $p_0(x)\,dx$.

By the Bayes formula

$$\mathbf{E}(f(X_t) \mid \mathcal{Y}_t) = \frac{\mathbf{E}^\dagger(f(X_t)Z_t \mid \mathcal{Y}_t)}{\mathbf{E}^\dagger(Z_t \mid \mathcal{Y}_t)}\ .$$

In addition

$$\mathbf{E}^\dagger(f(X_t)Z_t \mid \mathcal{Y}_t) = \int f(x)p_t(x)\,dx\ ,$$

where the unnormalized conditional density $\{p_t\,,\,t \geq 0\}$ satisfies the Zakai equation

$$dp_t = L^* p_t\,dt + B_k^* p_t\,dY_t^k\ , \tag{2.2}$$

see [15,17].

Consider then the following decomposition of the Zakai equation (2.2)

$$dp_t = L_0^* p_t\,dt + \Lambda^* p_t\,dt + B_k^* p_t\,dY_t^k\ ,$$

where

$$\Lambda \triangleq L - L_0 = \tfrac{1}{2}\sum_{i,j=1}^m c^{ij}\frac{\partial^2}{\partial x_i \partial x_j}\ .$$

On one hand, the partial differential operator L_0 generates a semi-group $\{P_t\,,\,t \geq 0\}$. On the other hand, it is possible to associate a stochastic semi-group $\{Q_t^s\,,\,0 \leq s \leq t\}$ with the following degenerate second-order stochastic PDE

$$dq_t = \Lambda^* q_t\,dt + B_k^* q_t\,dY_t^k\ , \tag{2.3}$$

which will be studied below. Therefore, it is worth studying the following Trotter-like product formulas for approximating the original Zakai equation (2.2)

$$\begin{aligned} \overline{p}_{i+1} &= P_{\delta_i}^*\,Q_{t_{i+1}}^{t_i}\,\overline{p}_i\ ,\\ \overline{p}_{i+1} &= Q_{t_{i+1}}^{t_i}\,P_{\delta_i}^*\,\overline{p}_i\ , \end{aligned} \tag{2.4}$$

where $\delta_i \triangleq t_{i+1} - t_i$.

The main interest of such product formulas is that the original equation has been splitted into a second–order deterministic PDE (*prediction* step), and a degenerate second–order stochastic PDE (*correction* step). In the case of independent noises, this stochastic PDE would reduce to a zero–order equation, for which there would exist a straightforward explicit solution. In the case of correlated noises, a representation result is available by the method of stochastic characteristics (i.e. involving the stochastic flow of diffeomorphisms associated with a SDE driven by the observation process), see Krylov–Rozovskii [8].

Remark 2.1 Written in Stratonovich form, equation (2.3) is a first–order stochastic PDE. For such an equation, one can use the representation result of Kunita [9,11], and translate the stochastic characteristics equations from Stratonovich form back to Itô form, to recover the representation result of [8].

As a consequence of the above discussion, there will be two steps in designing the approximation to the original Zakai equation (2.2)

- first use a Trotter–like product formula,
- then approximate the solution of the degenerate second–order stochastic PDE, by approximating the stochastic flow of diffeomorphisms involved in the stochastic characteristics method of [8].

It will be proved that the first step can be achieved with a rate of convergence of order $O(\delta)$, whereas the rate of convergence for the second step (and *a fortiori* for the global approximation procedure) is of order $O(\sqrt{\delta})$ only, where $\delta \triangleq \max_{i\geq 0} \delta_i$.

3 Trotter–like product formula

For all $n \geq 0$, let H^n denote the space of real–valued Lebesgue–measurable functions on $\mathbf{R}^m$ whose generalized derivatives up to order n are square–integrable, with norm $\|\cdot\|_n$

$$\|u\|_n^2 \triangleq \sum_{|\alpha| \leq n} \|D^\alpha u\|_0^2 < \infty \ .$$

In addition, the following shorthand notations will be used throughout the paper: $|\cdot| \triangleq \|\cdot\|_0$ and $\|\cdot\| \triangleq \|\cdot\|_1$.

The beginning of this section is devoted to recall existence, uniqueness and regularity results for the Zakai equation (2.2) and the degeneate second–order stochastic partial differential equation (2.3).

Consider first the Zakai equation

$$dp_t = L^* p_t \, dt + B_k^* p_t \, dY_t^k \ . \tag{3.1}$$

The following existence, uniqueness and regularity result is proved in Krylov–Rozovskii [7] and Pardoux [15].

Theorem 3.1 *Let $n \geq 0$ be fixed. Assume that the coefficients satisfy*

- *a, c and ρ have bounded derivatives up to order $(n+1)$,*
- *b and h have bounded derivatives up to order n,*

and that the initial condition satisfies $p_0 \in H^n$.
Then equation (3.1) has a unique solution $p \in M^2(0,T\,;\,H^{n+1})$. In addition

- *$p \in L^2(\Omega\,;\,C([0,T]\,;\,H^n))$,*
- *the following estimate holds*

$$\mathbf{E}^\dagger[\sup_{0 \leq t \leq T} \|p_t\|_n^2] \leq C \|p_0\|_n^2 \ .$$

Remark 3.2 The uniform ellipticity of the coefficient a is not essential here. Actually, a slightly weaker theorem holds in the more general case where a is only semi–definite positive, see [8].

Consider now the degenerate second–order stochastic PDE (2.3). Although no coercivity hypothesis is satisfied, the following result is proved by Krylov–Rozovskii [8].

Theorem 3.3 *Let $n \geq 1$ be fixed. Assume that the coefficients satisfy*

- *c and ρ have bounded derivatives up to order $(n+1)$,*
- *h has bounded derivatives up to order n,*

and that the initial condition satisfies $p_0 \in H^n$.
Then equation (2.3) has a unique weak solution $q \in M^2(0,T\,;\,H^n)$. In addition

- *$q \in L^2(\Omega\,;\,C([0,T]\,;\,H^{n-1}))$,*
- *$q \in L^2(\Omega\,;\,C_w([0,T]\,;\,H^n))$,*
- *the following estimate holds*

$$\mathbf{E}^\dagger[\sup_{0\leq t\leq T}\|q_t\|_n^2] \leq C\|q_0\|_n^2 \ .$$

Remark 3.4 The condition $n \geq 1$ is only needed to prove the existence of a solution. If this point is already settled, then the rest of the theorem holds for $n=0$ as well.

□ Error estimate

The purpose here is to study the Trotter–like product formula introduced in (2.4).

Theorem 3.5 *Consider the following approximation scheme*

$$\overline{p}_{i+1} = P^*_{\delta_i}\, Q^{t_i}_{t_{i+1}}\, \overline{p}_i \ .$$

Assume that the coefficients satisfy

- *a, c and ρ have bounded derivatives up to order 3,*
- *b and h have bounded derivatives up to order 2,*

and that the initial condition satisfies $p_0 \in H^2$.
Then $\overline{p}_i$ approximates the solution p_{t_i} of the original Zakai equation (3.1) with a rate of convergence of order $O(\delta)$. Indeed

$$\left\{\mathbf{E}^\dagger|\overline{p}_i - p_{t_i}|^2\right\}^{1/2} \leq C\;\delta\;\|p_0\|_2 \ .$$

PROOF. Throughout the proof, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between H^1 and H^{-1}. Define $v_t \triangleq P^*_{t-s}\, Q^s_t\, \phi$ for ϕ smooth enough. Differentiating with respect to t gives

$$\begin{aligned}
dv_t &= L^*_0 v_t\, dt + P^*_{t-s}\, \left[\Lambda^* Q^s_t\, \phi\, dt + B^*_k Q^s_t\, \phi\, dY^k_t\right] \\
&= L^*_0 v_t\, dt + \Lambda^* v_t\, dt + B^*_k v_t\, dY^k_t \\
&\quad + [P^*_{t-s}\, \Lambda^* - \Lambda^*\, P^*_{t-s}]\, Q^s_t\, \phi\, dt + [P^*_{t-s}\, B^*_k - B^*_k\, P^*_{t-s}]\, Q^s_t\, \phi\, dY^k_t \\
&= L^* v_t\, dt + B^*_k v_t\, dY^k_t + f_t\, dt + g^k_t\, dY^k_t\ ,
\end{aligned}$$

where the perturbation terms are defined by

$$f_t \triangleq [P^*_{t-s}\, \Lambda^* - \Lambda^*\, P^*_{t-s}]\, Q^s_t\, \phi \qquad \text{and} \qquad g^k_t \triangleq [P^*_{t-s}\, B^*_k - B^*_k\, P^*_{t-s}]\, Q^s_t\, \phi\ ,$$

respectively. The difference $\varepsilon_t \triangleq v_t - p_t$ satisfies

$$d\varepsilon_t = L^* \varepsilon_t\, dt + B^*_k \varepsilon_t\, dY^k_t + f_t\, dt + g^k_t\, dY^k_t\ .$$

The identity of energy of [15] gives

$$\begin{aligned}
|\varepsilon_t|^2 &= |\varepsilon_s|^2 + 2\int_s^t \langle L^* \varepsilon_\tau + f_\tau, \varepsilon_\tau\rangle\, d\tau \\
&\quad + 2\int_s^t (B^*_k \varepsilon_\tau + g_\tau, \varepsilon_\tau)\, dY^k_\tau + \int_s^t |B^* \varepsilon_\tau + g_\tau|^2\, d\tau\ .
\end{aligned}$$

Using the coercivity property and standard estimates, gives

$$\begin{aligned}
\mathbf{E}^\dagger |\varepsilon_t|^2 &\leq \mathbf{E}^\dagger |\varepsilon_s|^2 + C\, \mathbf{E}^\dagger \int_s^t |\varepsilon_\tau|^2\, d\tau \\
&\quad + C'\, \mathbf{E}^\dagger \int_s^t \|f_\tau\|^2_{-1}\, d\tau + C''\, \mathbf{E}^\dagger \int_s^t |g_\tau|^2\, d\tau\ .
\end{aligned}$$

Assume that the following estimates hold

$$\mathbf{E}^\dagger \|f_\tau\|^2_{-1} \leq C\, (\tau - s)^2\, \exp\{C(\tau - s)\}\, \mathbf{E}^\dagger \|\phi\|^2_2\ , \tag{3.2}$$

$$\mathbf{E}^\dagger |g_\tau|^2 \leq C\, (\tau - s)^2\, \exp\{C(\tau - s)\}\, \mathbf{E}^\dagger \|\phi\|^2_2\ . \tag{3.3}$$

Then Gronwall's lemma would yield

$$\mathbf{E}^\dagger |\varepsilon_t|^2 \leq \left[\mathbf{E}^\dagger |\varepsilon_s|^2 + C\, (t - s)^3\, \mathbf{E}^\dagger \|\phi\|^2_2\right]\, \exp\{C(t - s)\}\ ,$$

provided $\phi \in H^2$. Now, it follows from the assumptions and from Theorem 3.1, that $\overline{p}_i \in L^2(\Omega\, ;\, H^2)$ for all i, so that setting $s = t_i$, $t = t_{i+1}$ and $\phi = \overline{p}_i$, gives

$$\mathbf{E}^\dagger |\overline{p}_{i+1} - p_{t_{i+1}}|^2 \leq \left[\mathbf{E}^\dagger |\overline{p}_i - p_{t_i}|^2 + C\, (t_{i+1} - t_i)^3\, \mathbf{E}^\dagger \|\overline{p}_i\|^2_2\right]\, \exp\{C(t_{i+1} - t_i)\}\ ,$$

and the result follows from the discrete Gronwall lemma. The end of the proof is devoted to proving estimates (3.2) and (3.3).

□ *Estimate (3.2)*

The following perturbation result

$$[P^*_{\tau-s}\,\Lambda^* - \Lambda^*\,P^*_{\tau-s}]\,u = \int_s^\tau P^*_{\tau-\tau'}\,[L_0^*\,\Lambda^* - \Lambda^*\,L_0^*]\,P^*_{\tau'-s} u\,d\tau'\ ,$$

holds for u smooth enough. It follows from the assumptions, that the partial differential operator $D \triangleq [L_0^*\,\Lambda^* - \Lambda^*\,L_0^*]$ satisfies

$$|(D\phi, \psi)| = |(\Lambda^*\phi, L_0\psi) - (L_0^*\phi, \Lambda\psi)| \le C\ \|\phi\|_2\ \|\psi\|_{-1}\ ,$$

for ϕ and ψ smooth enough, which means that D is bounded from H^2 to H^{-1}. In addition, $\{P_t^*\,,\,t \ge 0\}$ is a strongly continuous semi-group in both H^{-1} and H^2. Therefore

$$\begin{aligned}\|f_\tau\|_{-1} &\le \int_s^\tau \|P^*_{\tau-\tau'}\ D\ P^*_{\tau'-s}\ Q_t^s\phi\|_{-1}\,d\tau' \\ &\le C\ (\tau - s)\ \exp\{C(\tau - s)\}\ \|Q_t^s\phi\|_2\ .\end{aligned}$$

Then

$$\begin{aligned}\mathbf{E}^\dagger\|f_\tau\|_{-1}^2 &\le C\ (\tau - s)^2\ \exp\{C(\tau - s)\}\ \mathbf{E}^\dagger\|Q_t^s\phi\|_2^2 \\ &\le C\ (\tau - s)^2\ \exp\{C(\tau - s)\}\ \mathbf{E}^\dagger\|\phi\|_2^2\ .\end{aligned}$$

□ *Estimate (3.3)*

Similarly, the following perturbation result

$$[P^*_{\tau-s}\,B_k^* - B_k^*\,P^*_{\tau-s}]\,u = \int_s^\tau P^*_{\tau-\tau'}\,[L_0^*\,B_k^* - B_k^*\,L_0^*]\,P^*_{\tau'-s} u\,d\tau'\ ,$$

holds for u smooth enough. It follows from the assumptions, that the partial differential operator $D_k \triangleq [L_0^*\,B_k^* - B_k^*\,L_0^*]$ is bounded from H^2 to H^0. In addition, $\{P_t^*\,,\,t \ge 0\}$ is a strongly continuous semi-group in both H^0 and H^2. Therefore

$$\begin{aligned}|g_\tau^k| &\le \int_s^\tau |P^*_{\tau-\tau'}\ D_k\ P^*_{\tau'-s}\ Q_t^s\phi|\,d\tau' \\ &\le C\ (\tau - s)\ \exp\{C(\tau - s)\}\ \|Q_t^s\phi\|_2\ .\end{aligned}$$

Then

$$\begin{aligned}\mathbf{E}^\dagger|g_\tau|^2 &\le C\ (\tau - s)^2\ \exp\{C(\tau - s)\}\ \mathbf{E}^\dagger\|Q_t^s\phi\|_2^2 \\ &\le C\ (\tau - s)^2\ \exp\{C(\tau - s)\}\ \mathbf{E}^\dagger\|\phi\|_2^2\ . \qquad \square\end{aligned}$$

4 Stochastic characteristics

Going back to stochastic differential systems, the following result holds

Theorem 4.1 *Let $\xi_{s,t}(\cdot)$ be the stochastic flow associated with the forward SDE*

$$d\xi_t = \rho(\xi_t)\,[dY_t - h(\xi_t)\,dt]\ . \tag{4.1}$$

Assume that the coefficients h and ρ have bounded derivatives up to order $(n+1)$. Then $\xi_{s,t}(\cdot)$ is a C^n-diffeomorphism in $\mathbf{R}^m$.

Under the assumption that the coefficient ρ has bounded derivatives up to order 2, the inverse map $\xi_{s,t}^{-1}(\cdot)$ is given explicitly as the (backward) stochastic flow $\eta_{t,s}(\cdot)$ associated with the backward SDE

$$d\eta_t = \rho(\eta_t) \oplus [dY_t - h(\eta_t)\,dt] - \rho_0(\eta_t)\,dt\ , \tag{4.2}$$

with

$$\rho_0^i \triangleq \sum_{k=1}^{d}\sum_{j=1}^{m} \frac{\partial \rho_k^i}{\partial x_j}\rho_k^j\ , \qquad 1 \le i \le m\ .$$

The regularity of $\xi_{s,t}(\cdot)$ was first proved by Blagoveschenskii–Freidlin [3], whereas the rest of the theorem is proved in Kunita [10].

Proposition 4.2 *The Jacobian $J_{s,t}(x)$ (i.e. the determinant of the Jacobian matrix) of the diffeomorphism $\xi_{s,t}(\cdot)$ satisfies*

$$\begin{aligned} J_{s,t}(x) \triangleq \exp\Big\{ &\int_s^t \alpha^*(\xi_{s,\tau}(x))\,[dY_\tau - h(\xi_{s,\tau}(x))\,d\tau] \\ &- \int_s^t h_0(\xi_{s,\tau}(x))\,d\tau - \int_s^t \overline{\alpha}(\xi_{s,\tau}(x))\,d\tau \Big\} \end{aligned} \tag{4.3}$$

with

$$\alpha_k \triangleq \sum_{i=1}^{m} \frac{\partial \rho_k^i}{\partial x_i} = \mathrm{div}\rho_k\ , \qquad 1 \le k \le d$$

$$\overline{\alpha} \triangleq \tfrac{1}{2} \sum_{k=1}^{d}\sum_{i,j=1}^{m} \frac{\partial \rho_k^i}{\partial x_j}\frac{\partial \rho_k^j}{\partial x_i} \quad \textit{and} \quad h_0 \triangleq \sum_{k=1}^{d}\sum_{i=1}^{m} \frac{\partial h_k}{\partial x_i}\rho_k^i\ .$$

PROOF. Transform first the SDE (4.1) into Stratonovich form

$$d\xi_t = \rho(\xi_t) \circ [dY_t - h(\xi_t)\,dt] - \tfrac{1}{2}\rho_0(\xi_t)\,dt\ .$$

From the corresponding result for ordinary differential equations, it holds

$$d\log J_{s,t}(x) = \alpha^*(\xi_{s,t}(x)) \circ [dY_t - h(\xi_{s,t}(x))\,dt] - h_0(\xi_{s,t}(x))\,dt - \tfrac{1}{2}\mathrm{div}\rho_0(\xi_{s,t}(x))\,dt\ .$$

Transforming back to Itô form gives

$$d\log J_{s,t}(x) = \alpha^*(\xi_{s,t}(x))\,[dY_t - h(\xi_{s,t}(x))\,dt] - h_0(\xi_{s,t}(x))\,dt$$

$$-\tfrac{1}{2}\mathrm{div}\rho_0(\xi_{s,t}(x))\,dt + \tfrac{1}{2}\alpha_0(\xi_{s,t}(x))\,dt\ .$$

Now it holds

$$\mathrm{div}\rho_0 = \sum_{k=1}^{d}\sum_{i,j=1}^{m}\frac{\partial}{\partial x_i}\left(\frac{\partial \rho_k^i}{\partial x_j}\rho_k^j\right)\ ,$$

$$\alpha_0 \triangleq \sum_{k=1}^{d}\sum_{i=1}^{m}\frac{\partial \alpha_k}{\partial x_i}\rho_k^i = \sum_{k=1}^{d}\sum_{i,j=1}^{m}\frac{\partial^2 \rho_k^j}{\partial x_i \partial x_j}\rho_k^i\ ,$$

which finishes the proof. □

Remark 4.3 Note that $[J_{s,t}(\eta_{t,s}(x))]^{-1}$ is actually the Jacobian of the inverse diffeomorphism $\eta_{t,s}(\cdot)$.

Define

$$\Xi_{s,t}(x) \triangleq \exp\left\{\int_s^t h^*(\xi_{s,\tau}(x))\,dY_\tau - \tfrac{1}{2}\int_s^t |h(\xi_{s,\tau}(x))|^2\,d\tau\right\}\ , \tag{4.4}$$

and

$$\Theta_{s,t}(x) \triangleq \Xi_{s,t}(x)\ [J_{s,t}(x)]^{-1} = \exp\left\{\int_s^t h^*(\xi_{s,\tau}(x))\,dY_\tau - \tfrac{1}{2}\int_s^t |h(\xi_{s,\tau}(x))|^2\,d\tau\right.$$

$$\left. - \int_s^t \alpha^*(\xi_{s,\tau}(x))\,[dY_\tau - h(\xi_{s,\tau}(x))\,d\tau] + \int_s^t h_0(\xi_{s,\tau}(x))\,d\tau + \int_s^t \overline{\alpha}(\xi_{s,\tau}(x))\,d\tau\right\}\ .$$

Introduce the following *definition*

$$Q_t^s\phi(x) \triangleq \phi(\eta_{t,s}(x))\ \Theta_{s,t}(\eta_{t,s}(x))\ , \tag{4.5}$$

or equivalently

$$Q_t^s\phi(\xi_{s,t}(x)) = \phi(x)\ \Theta_{s,t}(x)\ .$$

where the same notation has been used as in the previous section. This will be justified by the Theorem 4.8 to be proved below.

Remark 4.4 Under the additional assumption that the coefficient ρ has bounded derivatives up to order 2, the Lemma 6.2 of [10, Chapter II] gives the following explicit expressions in terms of backward Itô stochastic integrals

$$\Xi_{s,t}(\eta_{t,s}(x)) = \exp\left\{\int_s^t h^*(\eta_{t,\tau}(x)) \circledast dY_\tau - \tfrac{1}{2}\int_s^t |h(\eta_{t,\tau}(x))|^2\,d\tau - \int_s^t h_0(\eta_{t,\tau}(x))\,d\tau\right\}\ ,$$

$$J_{s,t}(\eta_{t,s}(x)) = \exp\left\{\int_s^t \alpha^*(\eta_{t,\tau}(x)) \oplus [dY_\tau - h(\eta_{t,\tau}(x))\, d\tau] \right.$$
$$\left. - \int_s^t h_0(\eta_{t,\tau}(x))\, d\tau - \int_s^t \overline{\alpha}(\eta_{t,\tau}(x))\, d\tau - \int_s^t \alpha_0(\eta_{t,\tau}(x))\, d\tau \right\} ,$$

where the coefficients h_0 and α_0 have already been defined as

$$h_0 \triangleq \sum_{k=1}^d \sum_{i=1}^m \frac{\partial h_k}{\partial x_i} \rho_k^i \quad \text{and} \quad \alpha_0 \triangleq \sum_{k=1}^d \sum_{i=1}^m \frac{\partial \alpha_k}{\partial x_i} \rho_k^i = \sum_{k=1}^d \sum_{i,j=1}^m \frac{\partial^2 \rho_k^j}{\partial x_i \partial x_j} \rho_k^i .$$

Therefore

$$\Phi_{t,s}(x) \triangleq \Theta_{s,t}(\eta_{t,s}(x)) = \exp\left\{\int_s^t h^*(\eta_{t,\tau}(x)) \oplus dY_\tau - \tfrac{1}{2}\int_s^t |h(\eta_{t,\tau}(x))|^2\, d\tau \right.$$
$$\left. - \int_s^t \alpha^*(\eta_{t,\tau}(x)) \oplus [dY_\tau - h(\eta_{t,\tau}(x))\, d\tau] + \int_s^t \overline{\alpha}(\eta_{t,\tau}(x))\, d\tau + \int_s^t \alpha_0(\eta_{t,\tau}(x))\, d\tau \right\} . \tag{4.6}$$

Remark 4.5 If $\rho \equiv 0$, then $\xi_{s,t}(x) = x$ so that

$$Q_t^s \phi(x) = \phi(x)\ \exp\left\{h^*(x)\,(Y_t - Y_s) - \tfrac{1}{2}|h(x)|^2\,(t-s)\right\} ,$$

which is actually the explicit solution of the equation

$$dq_t = h_k q_t\, dY_t^k ,$$

with initial condition ϕ at time s. In this case, (2.4) reduces to the discretization schemes considered in [2] and [12].

First, the following *stability* result holds

Proposition 4.6 *Let $n \geq 0$ be fixed. Assume that the coefficients satisfy*

- *h and ρ have bounded derivatives up to order $(n+1)$.*

If $\phi \in H^n$, then $Q_t^s\phi$ is a random variable with values in H^n, and in addition

$$\left\{\mathbf{E}^\dagger \|Q_t^s \phi\|_n^2\right\}^{1/2} \leq C\ \|\phi\|_n .$$

PROOF. It is enough to prove the result for $n = 0$.
The change of variable $x = \eta_{t,s}(y)$ i.e. $y = \xi_{s,t}(x)$ gives

$$\mathbf{E}^\dagger |Q_t^s \phi|^2 = \mathbf{E}^\dagger \int [|\phi(\eta_{t,s}(y))|\ \Theta_{s,t}(\eta_{t,s}(y))]^2\, dy$$
$$= \int |\phi(x)|^2\, \mathbf{E}\left\{\Theta_{s,t}(x)\right\}\, dx ,$$

and the result follows from the estimate

$$\sup_{x \in \mathbf{R}^m} \mathbf{E}\{\Theta_{s,t}(x)\} \leq C . \qquad \square$$

Another property of the two-parameter stochastic semigroup $\{Q_t^s, 0 \leq s \leq t\}$ is provided by the following

Proposition 4.7 *Let $\{T_t, t \geq 0\}$ be the semigroup generated by*

$$\Lambda = \tfrac{1}{2} \sum_{i,j=1}^{m} c^{ij} \frac{\partial^2}{\partial x_i \partial x_j} .$$

Then

$$\mathbf{E}^{\dagger} Q_t^s = T_{t-s}^* .$$

PROOF. The same change of variable as in the proof of Proposition 4.6, gives

$$\begin{aligned}(\mathbf{E}^{\dagger}(Q_t^s \phi), f) &= \mathbf{E}^{\dagger} \int \phi(\eta_{t,s}(y))\, \Theta_{s,t}(\eta_{t,s}(y))\, f(y)\, dy \\ &= \int \phi(x)\, \mathbf{E}[f(\xi_{s,t}(x))]\, dx .\end{aligned}$$

Now, under the original probability measure P

$$d\xi_t = \rho(\xi_t)\, dV_t ,$$

where $\{V_t, t \geq 0\}$ is a Wiener process with covariance matrix I. Therefore

$$(\mathbf{E}^{\dagger}(Q_t^s \phi), f) = (\phi, T_{t-s} f) = (T_{t-s}^* \phi, f) . \qquad \square$$

The following representation result of the solution of equation (2.3) in terms of the *stochastic characteristics* $\eta_{t,s}(\cdot)$, is the stochastic counterpart of the usual method of characteristics for linear first–order PDE's. It has been proved by Krylov–Rozovskii [8] and Kunita [9,11].

Theorem 4.8 *Let $\{Q_t^s, s \leq t\}$ be defined by (4.5). Then, the unique solution of equation (2.3) satisfies*

$$q_t(x) = Q_t^s q_s(x) . \tag{4.7}$$

PROOF. The proof given below is essentially that of [8]. Introduce

$$\gamma_t^s \triangleq \exp\left\{ \int_s^t \phi_\tau^*\, dY_\tau - \tfrac{1}{2} \int_s^t |\phi_\tau|^2\, d\tau \right\} ,$$

where $\{\phi_\tau\,,\ s \leq \tau \leq t\}$ is deterministic.

It follows from the Itô formula that $\overline{q}_t \triangleq \mathbf{E}^\dagger(\gamma_t^s q_t)$ satisfies

$$\overline{q}_t' = \Lambda^* \overline{q}_t + B_k^* \overline{q}_t\, \phi_t^k\ , \tag{4.8}$$

with the initial condition $\overline{q}_s = \mathbf{E}^\dagger(q_s)$. On the other hand, define

$$\overline{w}_\tau(x) \triangleq \mathbf{E}^\phi \left[f(\xi_{\tau,t}(x))\ \exp\left\{ \int_\tau^t \phi_{\tau'}^*\, h(\xi_{\tau,\tau'}(x))\, d\tau' \right\} \right]\ ,$$

where under the probability P^ϕ

$$d\xi_t = \rho(\xi_t)\ [dV_t^\phi + \phi_t\, dt]\ ,$$

and $\{V_t^\phi\,,\ t \geq 0\}$ is a Wiener process with covariance matrix I. By the Feynmann–Kac formula, $\{\overline{w}_\tau\,,\ s \leq \tau \leq t\}$ satisfies a PDE which is dual to (4.8), so that $(\overline{q}_t, f) = (\overline{q}_s, \overline{w}_s)$.

Consider now the right–hand side in the representation result (4.7). Then

$$\gamma_t^s\ Q_t^s q_s(x) = q_s(\eta_{t,s}(x))\ \Xi_{s,t}^\phi(\eta_{t,s}(x))\ \exp\left\{ \int_s^t \phi_\tau^*\, h(\eta_{t,\tau}(x))\, d\tau \right\}\ [J_{s,t}(\eta_{t,s}(x))]^{-1}\ ,$$

with

$$\Xi_{s,t}^\phi(x) \triangleq \exp\left\{ \int_s^t [h(\xi_{s,\tau}(x)) + \phi_\tau]^*\, dY_\tau - \tfrac{1}{2} \int_s^t |h(\xi_{s,\tau}(x)) + \phi_\tau|^2\, d\tau \right\}\ .$$

Define next $\overline{v}_t \triangleq \mathbf{E}^\dagger(\gamma_t^s\ Q_t^s q_s)$. The Fubini theorem, the change of variable $x = \eta_{t,s}(y)$ and the Lemma 6.2 of [10, Chapter II] give

$$\begin{aligned}
(\overline{v}_t, f) &= \mathbf{E}^\dagger \int f(y)\ q_s(\eta_{t,s}(y))\ \Xi_{s,t}^\phi(\eta_{t,s}(y))\ \exp\left\{ \int_s^t \phi_\tau^*\, h(\eta_{t,\tau}(y))\, d\tau \right\}\ [J_{s,t}(\eta_{t,s}(y))]^{-1}\, dy \\
&= \mathbf{E}^\dagger \int f(\xi_{s,t}(x))\ q_s(x)\ \Xi_{s,t}^\phi(x)\ \exp\left\{ \int_s^t \phi_\tau^*\, h(\xi_{s,\tau}(x))\, d\tau \right\}\ dx \\
&= \int \overline{q}_s(x)\ \mathbf{E}^\dagger \left[f(\xi_{s,t}(x))\ \Xi_{s,t}^\phi(x)\ \exp\left\{ \int_s^t \phi_\tau^*\, h(\xi_{s,\tau}(x))\, d\tau \right\} \right]\ dx \\
&= \int \overline{q}_s(x)\ \mathbf{E}^\phi \left[f(\xi_{s,t}(x))\ \exp\left\{ \int_s^t \phi_\tau^*\, h(\xi_{s,\tau}(x))\, d\tau \right\} \right]\ dx = (\overline{q}_s, \overline{w}_s)\ .
\end{aligned}$$

Since $(\overline{q}_t, f) = (\overline{v}_t, f)$ for arbitrary test–function f and arbitrary $\{\phi_\tau\,,\ s \leq \tau \leq t\}$, this finishes the proof. □

5 Approximation of the stochastic characteristics

It has been proved in Section 4 that the stochastic semi-group $\{Q_t^s\,,\, 0 \le s \le t\}$ associated with the degenerate second-order stochastic PDE (2.3) satisfies

$$Q_t^s \phi(x) = \phi(\eta_{t,s}(x))\, \Theta_{s,t}(\eta_{t,s}(x)) \;, \tag{5.1}$$

where $\eta_{t,s}(\cdot)$ is the inverse of the stochastic flow of diffeomorphisms $\xi_{s,t}(\cdot)$ associated with the SDE (4.1). The purpose of this section is to investigate approximations of (5.1).

Considering $\eta_{t,s}(\cdot)$ as the stochastic flow of diffeomorphisms associated with the backward SDE (4.2), and rewriting (5.1) as

$$Q_t^s \phi(x) = \phi(\eta_{t,s}(x))\, \Phi_{t,s}(x) \;,$$

where $\Phi_{t,s}(x)$ has been defined in (4.6), it is natural to consider the following approximation

$$\overline{Q}_t^s \phi(x) \triangleq \phi(\overline{\eta}_{t,s}(x))\, \overline{\Phi}_{t,s}(x) \;, \tag{5.2}$$

where

$$\overline{\eta}_{t,s}(x) \triangleq x - \rho(x)\, [Y_t - Y_s - h(x)\,(t-s)] + \rho_0(x)\,(t-s) \;,$$

and

$$\begin{aligned} \overline{\Phi}_{t,s}(x) \;\triangleq\; \exp\Big\{ & h^*(x)\,(Y_t - Y_s) - \tfrac{1}{2}|h(x)|^2\,(t-s) - \alpha^*(x)\,[Y_t - Y_s - h(x)\,(t-s)] \\ & + \overline{\alpha}(x)\,(t-s) + \alpha_0(x)\,(t-s) \Big\} \;, \end{aligned}$$

are *computable* approximations of $\eta_{t,s}(x)$ and $\Phi_{t,s}(x)$ respectively, both depending only on the increments $(Y_t - Y_s)$.

Remark 5.1 One possible approach would be to approximate $\eta_{t,s}(\cdot)$ by the stochastic flow of diffeomorphisms associated with the ordinary differential equation obtained from (4.2) by replacing the observation sample–path $\{Y_t\,,\, 0 \le t \le T\}$ with some regular approximation, such as the Euler stepwise approximation or the polygonal interpolation. The numerical analysis of such an approximation should not be very difficult. However, the resulting approximation would not be *explicitly computable.*

The remaining of this section is devoted to studying the rate of convergence of this approximation. First, a *stability* result similar to Proposition 4.6 is needed. However, such a result can not be proved in the same way here, because $\overline{\eta}_{t,s}(\cdot)$ is *not* a diffeomorphism. Therefore, an additional assumption is introduced on the correlation coefficient ρ.

Assumption (A) *For some $0 \le r \le \min(m,d)$, there exist a $m \times r$ matrix p independent of x, and a $r \times d$ matrix $\overline{p}(x)$ with full rank r, such that $\rho(x) = p \cdot \overline{p}(x)$. In addition, it is assumed that the matrix $\overline{c} \triangleq \overline{p}\,\overline{p}^*$ is uniformly elliptic, i.e. $\overline{c}(x) \ge \gamma I$.*

Remark 5.2 The first part of Assumption (A) is equivalent to saying that the $m \times d$ matrix $\rho(x)$ has constant rank r, and the r-dimensional vector space range$(\rho(x))$ does not depend on x.

Proposition 5.3 *Let $n \geq 0$ be fixed. Assume that the coefficients satisfy*

- *ρ has bounded derivatives up to order $(n+2)$,*
- *h has bounded derivatives up to order n.*

If $\phi \in H^n$, then under Assumption (A), $\overline{Q}_t^s \phi$ is a random variable with values in H^n, and in addition

$$\left\{\mathbf{E}^\dagger \|\overline{Q}_t^s \phi\|_n^2\right\}^{1/2} \leq C \, \|\phi\|_n .$$

Remark 5.4 The proof of this proposition is given in the Appendix. Since $\overline{\eta}_{t,s}(\cdot)$ is not a diffeomorphism, one can not use a change of variable as in the proof of Proposition 4.6. Instead, one uses the fact that $\overline{\eta}_{t,s}(x)$ and $\overline{\Phi}_{t,s}(x)$ are very simple functions of the Gaussian random variable $(Y_t - Y_s)$.

Remark 5.5 The approximations $\overline{\eta}_{t,s}(\cdot)$ and $\overline{\Phi}_{t,s}(\cdot)$ are based on the explicit expressions for $\eta_{t,s}(\cdot)$ and $\Phi_{t,s}(\cdot)$, given in (4.2) and (4.6) respectively. This explains why the regularity assumptions on the coefficients ρ and h are different in Proposition 4.6 and Proposition 5.3.

Next, the following proposition provides error estimate on commuting the operator $\overline{Q}_t^s$ and spatial derivatives.

Proposition 5.6 *Let $n \geq 0$ and α a multi-index, be fixed. Assume that the coefficients satisfy*

- *ρ has bounded derivatives up to order $(n + |\alpha| + 2)$,*
- *h has bounded derivatives up to order $(n + |\alpha|)$.*

If $\phi \in H^{n+|\alpha|}$, then under Assumption (A)

$$\left\{\mathbf{E}^\dagger \|\overline{Q}_t^s D^\alpha \phi - D^\alpha \overline{Q}_t^s \phi\|_n^2\right\}^{1/2} \leq C \, \sqrt{t-s} \, \|\phi\|_{n+|\alpha|} .$$

Here again, the proof of this proposition is given in the Appendix.

□ Overall error estimate

The main result of the paper is provided by the following

Theorem 5.7 *Consider the following approximation scheme*

$$\overline{\overline{p}}_{i+1} = P^*_{\delta_i} \, \overline{Q}^{t_i}_{t_{i+1}} \, \overline{\overline{p}}_i \; .$$

Assume that the coefficients satisfy

- *a and c have bounded derivatives up to order 4,*
- *b and ρ have bounded derivatives up to order 3,*
- *h has bounded derivatives up to order 2,*

and that the initial condition satisfies $p_0 \in H^2$.
Then $\overline{\overline{p}}_i$ approximates the solution p_{t_i} of the original equation (3.1) with a rate of convergence of order $O(\sqrt{\delta})$. Indeed

$$\left\{ \mathbf{E}^\dagger |\overline{\overline{p}}_i - p_{t_i}|^2 \right\}^{1/2} \leq C \, \sqrt{\delta} \; \|p_0\|_2 \; .$$

Proof. In view of Theorem 3.5, it is enough to prove that

$$\left\{ \mathbf{E}^\dagger |\overline{\overline{p}}_i - \overline{p}_i|^2 \right\}^{1/2} \leq C \, \sqrt{\delta} \; \|p_0\|_2 \; .$$

Let ϕ be smooth enough. Differentiating both sides of (5.2) with respect to t gives

$$\begin{aligned}
d\overline{Q}^s_t \phi(x) &= \frac{\partial \phi}{\partial x_i}(\overline{\eta}_{t,s}(x)) \; \Big[- \rho^i_k(x) \, [dY^k_t - h_k(x) \, dt] + \rho^i_0(x) \, dt \Big] \; \overline{\Phi}_{t,s}(x) \\
&\quad + \tfrac{1}{2} \frac{\partial^2 \phi}{\partial x_i \partial x_j}(\overline{\eta}_{t,s}(x)) \; [\rho^i_k(x) \rho^j_k(x) \, dt] \; \overline{\Phi}_{t,s}(x) \\
&\quad + \phi(\overline{\eta}_{t,s}(x)) \; \Big[h_k(x) \, dY^k_t - \tfrac{1}{2} |h(x)|^2 \, dt - \alpha_k(x) \; [dY^k_t - h_k(x) \, dt] \\
&\quad + \overline{\alpha}(x) \, dt + \alpha_0(x) \, dt + \tfrac{1}{2} |h(x) - \alpha(x)|^2 \, dt \Big] \; \overline{\Phi}_{t,s}(x) \\
&\quad + \frac{\partial \phi}{\partial x_i}(\overline{\eta}_{t,s}(x)) \; \Big[- \rho^i_k(x) \; [h_k(x) - \alpha_k(x)] \, dt \Big] \; \overline{\Phi}_{t,s}(x) \\
&= \tfrac{1}{2} c^{i,j}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(\overline{\eta}_{t,s}(x)) \; \overline{\Phi}_{t,s}(x) \, dt
\end{aligned}$$

$$+ \left[\rho_0^i(x) + \rho_k^i(x)\alpha_k(x)\right] \frac{\partial \phi}{\partial x_i}(\overline{\eta}_{t,s}(x))\, \overline{\Phi}_{t,s}(x)\, dt$$

$$+ \left[\overline{\alpha}(x) + \alpha_0(x) + \tfrac{1}{2}|\alpha(x)|^2\right] \phi(\overline{\eta}_{t,s}(x))\, \overline{\Phi}_{t,s}(x)\, dt$$

$$+ \left[h_k(x) - \alpha_k(x)\right] \phi(\overline{\eta}_{t,s}(x))\, \overline{\Phi}_{t,s}(x)\, dY_t^k - \rho_k^i(x) \frac{\partial \phi}{\partial x_i}(\overline{\eta}_{t,s}(x))\, \overline{\Phi}_{t,s}(x)\, dY_t^k \ .$$

Now, it can be checked that

$$\overline{\alpha} + \alpha_0 + \tfrac{1}{2}|\alpha|^2 = \tfrac{1}{2} \frac{\partial^2 c^{i,j}}{\partial x_i \partial x_j} \quad \text{and} \quad \rho_0^i + \rho_k^i \alpha_k = \frac{\partial c^{i,j}}{\partial x_j} \ .$$

Therefore, it holds

$$\begin{aligned} d\overline{Q}_t^s \phi(x) &= \Bigg[\tfrac{1}{2} c^{i,j}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(\overline{\eta}_{t,s}(x)) \\ &\quad + \frac{\partial c^{i,j}}{\partial x_j}(x)\, \frac{\partial \phi}{\partial x_i}(\overline{\eta}_{t,s}(x)) + \tfrac{1}{2} \frac{\partial^2 c^{i,j}}{\partial x_i \partial x_j}(x)\, \phi(\overline{\eta}_{t,s}(x))\Bigg]\, \overline{\Phi}_{t,s}(x)\, dt \\ &\quad + \left[h_k(x) \phi(\overline{\eta}_{t,s}(x)) - \rho_k^i(x)\, \frac{\partial \phi}{\partial x_i}(\overline{\eta}_{t,s}(x)) \ - \frac{\partial \rho_k^i}{\partial x_i}(x)\, \phi(\overline{\eta}_{t,s}(x))\right]\, \overline{\Phi}_{t,s}(x)\, dY_t^k \\ &= \left[\tfrac{1}{2} c^{i,j}(x)\, \overline{Q}_t^s \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) + \frac{\partial c^{i,j}}{\partial x_j}(x)\, \overline{Q}_t^s \frac{\partial \phi}{\partial x_i}(x) + \tfrac{1}{2} \frac{\partial^2 c^{i,j}}{\partial x_i \partial x_j}(x)\, \overline{Q}_t^s \phi(x)\right] dt \\ &\quad + \left[h_k(x)\, \overline{Q}_t^s \phi(x) - \rho_k^i(x)\, \overline{Q}_t^s \frac{\partial \phi}{\partial x_i}(x) - \frac{\partial \rho_k^i}{\partial x_i}(x)\, \overline{Q}_t^s \phi(x)\right] dY_t^k \end{aligned}$$

so that

$$\begin{aligned} d\overline{Q}_t^s \phi &= \Lambda^* \overline{Q}_t^s \phi\, dt + B_k^* \overline{Q}_t^s \phi\, dY_t^k \\ &\quad + \tfrac{1}{2} c^{i,j} \left[\overline{Q}_t^s \frac{\partial^2 \phi}{\partial x_i \partial x_j} - \frac{\partial^2}{\partial x_i \partial x_j} \overline{Q}_t^s \phi\right] dt + \frac{\partial c^{i,j}}{\partial x_j} \left[\overline{Q}_t^s \frac{\partial \phi}{\partial x_i} - \frac{\partial}{\partial x_i} \overline{Q}_t^s \phi\right] dt \\ &\quad - \rho_k^i \left[\overline{Q}_t^s \frac{\partial \phi}{\partial x_i} - \frac{\partial}{\partial x_i} \overline{Q}_t^s \phi\right] dY_t^k \ . \end{aligned}$$

The difference $\varepsilon_t \stackrel{\Delta}{=} \overline{Q}_t^s \phi - Q_t^s \psi$ satisfies

$$d\varepsilon_t = \Lambda^* \varepsilon_t\, dt + B_k^* \varepsilon_t\, dY_t^k + f_t\, dt + g_t^k\, dY_t^k \ ,$$

where the perturbation terms are defined by

$$f_t \stackrel{\Delta}{=} \tfrac{1}{2} c^{i,j} \left[\overline{Q}_t^s \frac{\partial^2 \phi}{\partial x_i \partial x_j} - \frac{\partial^2}{\partial x_i \partial x_j} \overline{Q}_t^s \phi\right] + \frac{\partial c^{i,j}}{\partial x_j} \left[\overline{Q}_t^s \frac{\partial \phi}{\partial x_i} - \frac{\partial}{\partial x_i} \overline{Q}_t^s \phi\right] \ ,$$

and

$$g_t^k \triangleq -\rho_k^i \left[\overline{Q}_t^s \frac{\partial \phi}{\partial x_i} - \frac{\partial}{\partial x_i} \overline{Q}_t^s \phi\right] ,$$

respectively. The identity of energy of [15] gives

$$|\varepsilon_t|^2 = |\varepsilon_s|^2 + 2\int_s^t \langle \Lambda^* \varepsilon_\tau, \varepsilon_\tau \rangle \, d\tau + 2\int_s^t (f_\tau, \varepsilon_\tau) \, d\tau$$

$$+ 2\int_s^t (B_k^* \varepsilon_\tau + g_\tau, \varepsilon_\tau) \, dY_\tau^k + \int_s^t |B^* \varepsilon_\tau + g_\tau|^2 \, d\tau .$$

Although no coercivity property exists, standard estimates give

$$\mathbf{E}^\dagger |\varepsilon_t|^2 \leq \mathbf{E}^\dagger |\varepsilon_s|^2 + C \, \mathbf{E}^\dagger \int_s^t |\varepsilon_\tau|^2 \, d\tau$$

$$+ C' \, \mathbf{E}^\dagger \int_s^t |f_\tau|^2 \, d\tau + C'' \, \mathbf{E}^\dagger \int_s^t |g_\tau|^2 \, d\tau .$$

Moreover, it follows from Proposition 5.6 that

$$\mathbf{E}^\dagger |f_\tau|^2 \leq C \, (\tau - s)^2 \, \exp\{C(\tau - s)\} \, \mathbf{E}^\dagger \|\phi\|_2^2 ,$$

$$\mathbf{E}^\dagger |g_\tau|^2 \leq C \, (\tau - s)^2 \, \exp\{C(\tau - s)\} \, \mathbf{E}^\dagger \|\phi\|^2 ,$$

and therefore Gronwall's lemma would yield

$$\mathbf{E}^\dagger |\varepsilon_t|^2 \leq \left[\mathbf{E}^\dagger |\varepsilon_s|^2 + C \, (t - s)^2 \, \mathbf{E}^\dagger \|\phi\|_2^2\right] \, \exp\{C(t - s)\} ,$$

provided $\phi \in H^2$. Now, it follows from the assumptions and from Proposition 5.3, that $\overline{\overline{p}}_i \in L^2(\Omega\, ; \, H^2)$ for all i, so that setting $s = t_i$, $t = t_{i+1}$, $\phi = \overline{\overline{p}}_i$ and $\psi = \overline{p}_i$, gives

$$\mathbf{E}^\dagger |\overline{Q}_{t_{i+1}}^{t_i} \overline{\overline{p}}_i - Q_{t_{i+1}}^{t_i} \overline{p}_i|^2 \leq \left[\mathbf{E}^\dagger |\overline{\overline{p}}_i - \overline{p}_i|^2 + C \, (t_{i+1} - t_i)^2 \, \mathbf{E}^\dagger \|\overline{\overline{p}}_i\|_2^2\right] \, \exp\{C(t_{i+1} - t_i)\} .$$

Next

$$\mathbf{E}^\dagger |\overline{\overline{p}}_{i+1} - \overline{p}_{i+1}|^2 = \mathbf{E}^\dagger | \, P_{\delta_i}^* [\overline{Q}_{t_{i+1}}^{t_i} \overline{\overline{p}}_i - Q_{t_{i+1}}^{t_i} \overline{p}_i \,] |^2$$

$$\leq \left[\mathbf{E}^\dagger |\overline{\overline{p}}_i - \overline{p}_i|^2 + C \, (t_{i+1} - t_i)^2 \, \mathbf{E}^\dagger \|\overline{\overline{p}}_i\|_2^2\right] \, \exp\{C(t_{i+1} - t_i)\} ,$$

and the result follows from the discrete Gronwall lemma. □

Remark 5.8 The purpose of the Assumption (A) is only to provide a sufficient condition for the stability estimate to hold.

A further step in the time–discretization would consist in approximating the Fokker–Planck semi–group $\{P_t^* \, , \, t \geq 0\}$, using some classical approximation scheme. For instance, using the backward Euler scheme would result in the following global approximation scheme

$$(I - \delta_i \, L_0^*) \, \overline{\overline{p}}_{i+1} = \overline{Q}_{t_{i+1}}^{t_i} \, \overline{\overline{p}}_i ,$$

with same error estimate.

□ Particle approximation

Another possible approach to approximate the degenerate second-order stochastic PDE (2.3) — based also on the representation (5.1) in terms of stochastic characteristics — would be to use *particle methods*, adapting the results presented in Raviart [16] for deterministic first-order PDE's. The basic idea is to solve exactly equation (2.3) for an approximation of the initial condition, rather than approximate the stochastic characteristics as was done before.

Suppose that, at time t_i an approximation of the conditional probability law $\phi(x)\,dx$ is available, due e.g. to space discretization, in terms of a convex linear combination of Dirac masses sitting at some particle locations $\{x_i^k\,,\ k \in K\}$ with corresponding weights $\{a_i^k\,,\ k \in K\}$ i.e.

$$\phi(x)\,dx \sim \sum_{k \in K} a_i^k\,\delta(x - x_i^k)\ . \tag{5.3}$$

Solving exactly equation (2.3) in weak sense, with the approximation (5.3) as initial condition, gives the following approximation

$$Q_{t_{i+1}}^{t_i}\,\phi(x)\,dx \sim \sum_{k \in K} a_{i+1}^k\,\delta(x - x_{i+1}^k)$$

at time t_{i+1}. The new particle locations $\{x_{i+1}^k\,,\ k \in K\}$ and the corresponding weights $\{a_{i+1}^k\,,\ k \in K\}$ are computed according to

$$x_{i+1}^k = \xi_{t_i,t_{i+1}}(x_i^k) \quad \text{and} \quad a_{i+1}^k = a_i^k\ \Xi_{t_i,t_{i+1}}(x_i^k)\ ,$$

where $\xi_{s,t}(\cdot)$ is the diffeomorphism associated with equation (4.1), and $\Xi_{s,t}(\cdot)$ has been defined in (4.4).

The error estimate associated with this particle approximation will be studied elsewhere.

6 Conclusion

A time–discretization scheme of the Zakai equation for diffusion processes observed in correlated noise has been proposed, based on the stochastic characteristics introduced in [8,9,11]. Under an additional assumption on the correlation coefficient, it has been shown that the rate of convergence of this approximation is of order $\sqrt{\delta}$, where δ is the time discretization step.

The same rate of convergence has been obtained in Elliott–Glowinski [5] for a different approximation

- on one hand, the approximation considered in [5] has a probabilistic interpretation, which is not the case for the time discretization scheme presented here,
- on the other hand, the latter is *actually computable*, whereas no numerical algorithm is provide to *compute* the approximation considered in [5].

Another point of interest would be to study some particle approximation for the degenerate second–order stochastic PDE, adapting the results presented in Raviart [16] for deterministic first–order PDE's.

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A Stability and commutation estimates

The purpose of this appendix is to prove the stability and commutation estimates for the approximation introduced in Section 5.

PROOF OF PROPOSITION 5.3. It is enough to prove the result for $n = 0$.
Using the variables (x, v) with $v = w - h(x)\,(t-s)$, gives

$$
\begin{aligned}
\mathbf{E}^\dagger |\overline{Q}_t^s \phi|^2 &= \mathbf{E}^\dagger \int \left[\, |\phi(\overline{\eta}_{t,s}(x))|\, \overline{\Phi}_{t,s}(x) \right]^2 dx \\
&= \frac{C}{(t-s)^{d/2}} \iint |\phi(x - \rho(x)\,[w - h(x)\,(t-s)] + \rho_0(x)\,(t-s))|^2 \\
&\qquad \exp\left\{2h^*(x)\,w - |h(x)|^2\,(t-s) - 2\alpha^*(x)\,[w - h(x)\,(t-s)]\right. \\
&\qquad \left. + 2\overline{\alpha}(x)\,(t-s) + 2\alpha_0(x)\,(t-s)\right\} \exp\left\{- \frac{|w|^2}{2(t-s)}\right\} dw\, dx \\
&= \frac{C}{(t-s)^{d/2}} \iint |\phi(x - \rho(x)\,v + \rho_0(x)\,(t-s))|^2 \\
&\qquad \exp\left\{2h^*(x)\,v + |h(x)|^2\,(t-s) - 2\alpha^*(x)\,v\right. \\
&\qquad \left. + 2\overline{\alpha}(x)\,(t-s) + 2\alpha_0(x)\,(t-s)\right\} \exp\left\{- \frac{|v + h(x)\,(t-s)|^2}{2(t-s)}\right\} dv\, dx \\
&= \frac{C}{(t-s)^{d/2}} \iint |\phi(x - \rho(x)\,v + \rho_0(x)\,(t-s))|^2 \\
&\qquad \exp\left\{h^*(x)\,v + \tfrac{1}{2}|h(x)|^2\,(t-s) - 2\alpha^*(x)\,v\right. \\
&\qquad \left. + 2\overline{\alpha}(x)\,(t-s) + 2\alpha_0(x)\,(t-s)\right\} \exp\left\{- \frac{|v|^2}{2(t-s)}\right\} dv\, dx\ .
\end{aligned}
$$

From the Young inequality

$$
2\,[h(x) - 2\alpha(x)]^*\,v \le A|h(x) - 2\alpha(x)|^2\,(t-s) + \frac{|v|^2}{A(t-s)}\ ,
$$

it follows that

$$
\mathbf{E}^\dagger |\overline{Q}_t^s \phi|^2 \le \frac{C}{(t-s)^{d/2}} \iint |\phi(x - \rho(x)\,v + \rho_0(x)\,(t-s))|^2 \exp\left\{- \frac{|v|^2}{2A'(t-s)}\right\} dv\, dx\ ,
$$

where A and A' are conjugate. Using the Assumption (A) and the variables (x, z) with $z = \overline{\rho}(x)\, v$, gives

$$\begin{aligned}
\mathbf{E}^{\dagger}|\overline{Q}_t^s\phi|^2 &\leq \frac{C}{(t-s)^{d/2}} \iint |\phi(x - pz + \rho_0(x)\,(t-s))|^2 \\
&\qquad \frac{1}{\sqrt{\det \overline{c}(x)}} \exp\Big\{-\frac{z^*\overline{c}^{-1}(x)z}{2A'(t-s)}\Big\}\, dz\, dx \\
&\leq \frac{C}{\sqrt{\gamma}(t-s)^{d/2}} \iint |\phi(x - pz + \rho_0(x)\,(t-s))|^2 \exp\Big\{-\frac{|z|^2}{2CA'(t-s)}\Big\}\, dz\, dx\,.
\end{aligned}$$

The application $F(x) \triangleq x - pz + \rho_0(x)\,(t-s)$ is a diffeomorphism provided $0 \leq t-s < 1/C$, and moreover the Jacobian of $F(x)$ is bounded below by $(1 - C(t-s))$. Therefore, using the variables (y, z) with $y = F(x)$, gives

$$\begin{aligned}
\mathbf{E}^{\dagger}|\overline{Q}_t^s\phi|^2 &\leq \frac{C}{(t-s)^{d/2}} \iint |\phi(y)|^2 \exp\Big\{-\frac{|z|^2}{2CA'(t-s)}\Big\}\, dz\, dy \\
&\leq C \int |\phi(y)|^2\, dy = C\, |\phi|^2\,,
\end{aligned}$$

provided $0 \leq t - s \leq \delta < 1/C$, which finishes the proof. □

PROOF OF PROPOSITION 5.6. Here again, it is enough to prove the result for $n = 0$ and $|\alpha| = 1$.

For ϕ smooth enough, it holds

$$\begin{aligned}
\frac{\partial}{\partial x_i}\overline{Q}_t^s\phi(x) &= \frac{\partial \phi}{\partial x_j}(\overline{\eta}_{t,s}(x))\, \Big[\delta^{ij} - \frac{\partial \rho_k^j}{\partial x_i}(x)\, [Y_t^k - Y_s^k - (t-s)\, h_k(x)] \\
&\qquad + (t-s)\, \Big(\rho_k^j(x)\, \frac{\partial h_k}{\partial x_i}(x) + \frac{\partial \rho_0^j}{\partial x_i}(x)\Big)\Big]\; \overline{\Phi}_{t,s}(x) \\
&\qquad + \phi(\overline{\eta}_{t,s}(x))\, \Big[\Big(\frac{\partial h_k}{\partial x_i}(x) - \frac{\partial \alpha_k}{\partial x_i}(x)\Big)\; [Y_t^k - Y_s^k - (t-s)\, h_k(x)] \\
&\qquad + (t-s)\, \Big(\alpha_k(x)\, \frac{\partial h_k}{\partial x_i}(x) + \frac{\partial \overline{\alpha}}{\partial x_i}(x) + \frac{\partial \alpha_0}{\partial x_i}(x)\Big)\Big]\; \overline{\Phi}_{t,s}(x) \\
&= \overline{Q}_t^s \frac{\partial \phi}{\partial x_i}(x) - \frac{\partial \rho_k^j}{\partial x_i}(x)\, [Y_t^k - Y_s^k - (t-s)\, h_k(x)]\, \overline{Q}_t^s \frac{\partial \phi}{\partial x_j}(x) \\
&\qquad + (t-s)\, \Big(\rho_k^j(x)\, \frac{\partial h_k}{\partial x_i}(x) + \frac{\partial \rho_0^j}{\partial x_i}(x)\Big)\; \overline{Q}_t^s \frac{\partial \phi}{\partial x_j}(x)
\end{aligned}$$

$$+ \left(\frac{\partial h_k}{\partial x_i}(x) - \frac{\partial \alpha_k}{\partial x_i}(x) \right) \; [Y_t^k - Y_s^k - (t-s)\, h_k(x)] \; \overline{Q}_t^s \phi(x)$$

$$+ (t-s) \left(\alpha_k(x) \, \frac{\partial h_k}{\partial x_i}(x) + \frac{\partial \overline{\alpha}}{\partial x_i}(x) + \frac{\partial \alpha_0}{\partial x_i}(x) \right) \; \overline{Q}_t^s \phi(x) \; .$$

Therefore, using the stability estimate provided by Proposition 5.3

$$\begin{aligned} \mathbf{E}^\dagger \left| \frac{\partial}{\partial x_i} \overline{Q}_t^s \phi - \overline{Q}_t^s \frac{\partial \phi}{\partial x_i} \right|^2 &\leq C\,(t-s) \; \left[\sum_{j=1}^m \mathbf{E}^\dagger \left| \overline{Q}_t^s \frac{\partial \phi}{\partial x_j} \right|^2 + \mathbf{E}^\dagger |\overline{Q}_t^s \phi|^2 \right] \\ &\leq C\,(t-s) \; \left[\sum_{j=1}^m \left| \frac{\partial \phi}{\partial x_j} \right|^2 + |\phi|^2 \right] \\ &\leq C\,(t-s) \; \|\phi\|^2 \; . \qquad \Box \end{aligned}$$

Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique

ISSN 0249-6399